Extreme events and entropy: a generalized quantile utility model

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Abstract

This paper extends Cumulative Prospect Theory by introducing a quantile representation in an ambiguous setting where the DM has multiple priors. We show a representation theorem in which an ambiguous act is valued by a functional defined through quantiles. The new functional is able to represent not only asymmetric attitude with respect to ambiguity on extreme events (optimism with respect to windfall gains and pessimism with respect to catastrophic events) but also the DM's attitude to consider entropy as a rule of inference when information is ambiguous and scant.

Keywords: Ambiguity, Entropy, Extreme Events, Multiple Priors, Quantiles.

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1 Introduction

In previous papers, Basili *et al.* (2005; 2008) show characterizations of the decision-maker's (DM henceforth) behavior when she reveals ambiguity attitude with respect to catastrophic losses or windfall gains and ambiguity neutrality with respect to a set of ordinary outcomes that forms the reference set. Both papers are consistent with the experimental literature (Camerer, 1989; Starmer, 2000; Wakker, 2006; de Lara Resende and Wu, 2010) on Prospect Theory (PT) and Cumulative Prospect Theory (CPT) that confirms the inverse S-shape of the Probability Weighting Function (PWF). PT and CPT show the existence of a positive correlation between the DM's attitude and the unusual size of the outcomes, even if they assume the existence of a sure and unique reference point. In the last few years the critical assumption of one certain reference point has been relaxed to cope with some real observed phenomena such as Preference Reversal or Counterfactual Thinking[1], even if the former phenomenon has been explained in models of context-free preferences, in which at least one of the axioms of expected utility theory has been relaxed: independence and/or reduction axioms (Karni and Safra, 1987; Segal, 1988) or transitivity of preferences (Loomes, Starmer and Sugden, 1989; Humphrey, 2001).

Crucially to the aim of this paper, Preference Reversal and Counterfactual Thinking have been described in variants of prospect theory by introducing reference point as lotteries (introducing reference point as lotteries (Koszegi and Rabin, 2006; Schmidt, Starmer and Sugden, 2008) or more simply recognizing the possible use of multiple reference points in judgments of post-decisional regret (Boles and Messick, 1995; Tsiros 1998; Lin, Huamg and Zeelenberg, 2006). Differently from those versions of CPT that generalize the reference-dependent subjective expected utility by allowing the reference point to be a lottery or simply a set of a few reference points, we introduce a new version of the decision-making rule under ambiguity based on quantiles on outcomes, without any reference to the shape of the DM's utility function. In fact similar to the characterization of the decision rule in Basili *et al.* (2005; 2008), our quantile representation involves only a ranking among all outcomes and disregards any consideration of preferences.

Quantile Utility Models (QUM) have been used in decision theory by Manski (1988) and axiomatized by Rostek (2010), who provides a characterization in a Savage setting; whereas Chambers (2007) axiomatizes a quantile representation of aggregation of multiple utilities in a social utility functional and characterizes quantiles on the space of distribution functions, by assuming monotonicity and ordinal covariance (2009).

In a risky set-up, a QUM can be summarized as follows: given a probability measure on the measurable space of outcomes and a fixed number $\gamma \in (0,1)$, the DM orders feasible alternatives with respect to the highest γ^{th}-*quantile* of the induced cumulative probability distribution over outcomes. Given a lottery,

[1] Counterfactual Thinking involves the psychological process of comparing the obtained outcome with other possible outcomes, which has become known as counterfactual (Roese, 1997).

the DM is assumed to maximize a fixed quantile of the utility probability distribution over outcomes[2], interestingly enough if $\gamma = 0$ or $\gamma = 1$, the optimal choice induced, respectively, by the standard maxmin or maxmax decisional rule is obtained. Crucially and unsurprisingly, this decisional rule has a clear limit in discriminating among probability distributions, since the DM values only a lottery by the fixed γ^{th}-*quantile* realization, i.e., a single statistic, whereas she is not interested in what happens in a probability distribution outside the γ^{th}-*quantile*, producing in such a way very large classes of indifference.

In this paper, we extend the CPT by introducing a quantile representation in a setting with a DM that has multiple priors on possible events. In such a way, we generalize the CPT and give an original characterization of a decision-making problem with multiple reference and extreme points. Following the standard literature, the set of priors might reflect the DM's assessment of the reliability of available information about the underlying uncertainty, that is, her perception of ambiguity, and optimal decision rules incorporate the DM's attitude about scanty and vague information. Moreover, this new representation allows us to elicit a new way of ranking alternatives, taking into account fat-tailed events that are usually misvalued in different approaches.

Let S be the set of states of the world and $\mathcal{A}$ be a σ-*algebra* of events where $\mathcal{A}$ is a subset of 2^S, taken to be equal to 2^S if S is finite. We consider a DM facing uncertainty, where uncertainty is modeled through the *core* of a convex capacity on $\mathcal{A}$, i.e., through the set $C(v)$ of a probability distributions P on $(S, \mathcal{A})$ above v, i.e., such that $P(A) \geq v(A)\ \forall A \in \mathcal{A}$.

In this paper, we mainly confine to $(S, 2^S)$ where S is finite; it is the object of future research to extend our results to an infinite setting. In fact, reasoning in a finite setting allows us to lean existing results (not extended to the infinite case, as far as we know), as the following first one, which consists of focusing on one particular probability P in $C(v)$ to be denoted π, the one that maximizes the entropy, which can be considered as the *less diffuse* in $P = C(v)$, and is known to dominate any probabilities in $C(v)$ for Lorenz ordering.[3]

Because of ambiguous information, the DM disentangles the problem of assigning values to probabilities distributions in $C(v)$ by referring, using the method of inference, to a decision rule that is intuitive and consistent, indeed the *Maximum Entropy Principle.*

The notion of entropy was introduced by Clausius (1864), who postulated that the entropy (a macrostate quantity derived from the second law of thermodynamics) of a closed system cannot decrease. Boltzmann defined the thermodynamical probability of a macrostate (particle), if all configurations are assumed to be equally likely, and he found that if the number of macrostates is

[2]Similarly, in a utility mass model, the DM maximizes "the probability of obtaining an outcome whose utility is higher than some fixed critical value" (Manski 1988, 80).

[3]The Lorenz order can be considered a classical majorization that induces a natural order on probability densities, and such a probability distribution $\pi \in C(v)$ is analogous to the unique egalitarian allocation in a convex game. As a matter of fact, the unique egalitarian allocation always exists in the core of a convex game, and Lorenz dominates every allocation in it (Dutta and Ray 1989).

very large, then the Sterling approximation of the multinomial coefficient of the number of microstates realizing the macrostate is equal to the entropy. In 1927, von Neumann associated an entropy quantity of a quantum state with a statistical operator (density operator) that describes the gas behavior. Later, Shannon (1948) introduced entropy as uncertainty measure or information measure, that is the average amount of information contained in a random variable.[4]In von Neumann's (1955) celebrated formula, entropy is defined in terms of the trace of the density matrix representing the state of particles of the gas.[5] The *Maximum Entropy Principle* was introduced by Jaynes (1957a, 1957b) to elicit the most unbiased or the most uniform distribution among all the possible ones in physics as a generalization of the classical principle of Laplace's (1824) Insufficient Reason.[6] In economics, Lorenz (1905) shows a simple method to give an ordered representation of income or wealth unequal distribution, which is known like the Lorenz Curve; indeed, one distribution is more uneven than another if it always dominates the others. The Lorenz order can be considered a classical majorization that induces a natural order on probability densities. The majorization ordering was defined by Muirhead (1903) and developed in application on symmetric means by Hardy, Littlewood and Polya (1928/29). Majorization determines the degree of similarity between the elements of two vectors, that is, it measures the degree to which the elements of two vectors differ (Ben-Haim and Dvorkind, 2004). It is evident that if entropy is interpreted as a measure of uniformity, then maximization of entropy means to reach the situation closer to uniformity, that it is the same of vectors majorization.[7]

Our goal in this paper is to translate in terms of attitude toward lower tails, upper tails and intermediate quantiles the model of Basili *et al.* (2008) where the DM was supposed to be pessimistic with respect to purely catastrophic losses, ambiguity neutral with respect to familiar outcomes and optimistic with respect to purely windfall gains.

So far for any given act $X : S \to \mathbb{R}$ and $(\alpha, \beta) \in [0,1]^2, \alpha \leq \beta$, such that

[4]As reported in Tribus and McIrvine, many years later, Shannon said: "My greatest concern was what to call it. I thought of calling it 'information', but the word was overly used, so I decided to call it 'uncertainty'. When I discussed it with John von Neumann, he had a better idea. von Neumann told me, 'you should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage".

[5]Given a statistical operator ρ (*transition matrix*) and the total number of particles (molecules) N, the von Neumann equation of entropy S is $S = -NTr(\rho \log \rho)$. Technically, $\log \rho$ is the Hermitian operator, which corresponds to orthogonal states of the quantum system, and it has exactly the same eigenvectors as ρ, which are considered the base 2 logarithm of the corresponding eigenvalues.

[6]"Jaynes used precisely the same formal procedure, but with a different interpretation, since he was concerned with the choice of distributions of nonphysical probabilities, sometimes called 'credibilities', as a method of replacing the use of a Bayes-Laplace postulate of a uniform distribution of credibility. He applied the method to statistical mechanics following earlier writers such as Boltzmann and Gibbs" (Good, 1963).

[7]Morris and Shin introduce a Laplacian belief to represent the uniform prior applied to unknown events. In global games, Laplacian belief represents the "diffuse or agnostic view on the actions of other players in the game" (Morris and Shin, 2003, 62).

$[\alpha, \beta]$ determines the interval of cumulative probability between which outcomes can be considered as ordinary (familiar), we assume (*Section 2*) that the DM values these outcomes between these two quantiles in an ambiguity neutral way with the help of π. We furthermore intend to model pessimism in the lower tail $[0, \alpha]$, i.e. to model the attitude of the DM that minimizes the 'expectation' of X on this probability interval with respect to all $P \in C(v)$, and symmetrically to model optimism in the upper tail $[\beta, 1]$, i.e., maximization of the 'expectation' of X in this 'probability interval' with respect to all P in $C(v)$.

For this purpose (*Section 3*), we need to introduce a new formula that fits the Choquet integral when expressed as an integral of an appropriate quantile function. While such a formula already exists in the literature (Denneberg 1994), we aim at proposing here a simpler direct expression, which might be of self-interest. Indeed, we obtain an inverse cumulative function that has a very attractive characteristic: it is a *platykurtic* (*fat-tailed*) *function*. In fact, it is the flattest (less peaked) in the interval $[\alpha, \beta]$, through maximization of entropy among all ones in $C(v)$, it has the fattest (the most divergent) loss tail among all ones in $C(v)$, since it maximizes the minimum expected value in the interval $[0, \alpha]$ and it has the fattest (the most divergent) gain tail among all ones in $C(v)$, since it maximizes the maximum expected value in the interval $[\beta, 1]$. Crucially, our composed inverse cumulative function might be amenable to fit a market evaluation or other performance of financial assets[8] and characterize an operational notion of the Precautionary Principle able to represent the impact of catastrophic events on human communities. We illustrate our model in an application (*Section 4*) where the a priori information leads to the uniform distribution P_0, but cautiousness forces the DM to envision an $\mathcal{E}$-*contamination* v of P_0; thus, we derive the less diffuse π in $C(v)$ and compute our proposed formula that performs this case. Finally, we define a simpler approach where the DM's set of beliefs can be represented by the $\mathcal{E}$-*contamination* of a given prior P_0, which is considered a credible probability even if not fully reliable, and the parameter $\mathcal{E}$ is the value that captures the confidence in the DM's assessment or the error in P_0. Concluding remarks follow.

2 Choquet integral and quantile-functions

Here we develop a simple generalization of quantile-function suited for an easy computation of the Choquet integral. This is performed in a finite setting for easier exposition, but generalizes rather easily to an infinite setting.

Let $(S, 2^S)$, where S is finite, and P probability on $(S, \mathcal{A} = 2^S)$. For $X : S \to \mathbb{R}$, the cumulative distribution function F_X of X is defined by $x \in \mathbb{R} \to F_X(x) = P(X \leq x)$[9]. A common pseudo-inverse of F_X denoted the quantile-

[8] Onour (2010) finds that the estimated values of the right and left tails' parameters in the Kuwait, Saudi and Dubai stock returns indicate that extreme losses and gains are both significant and have high chances of occurring.

[9] To avoid confusion, it will be often useful here to write F_X^P instead of F_X

function $F_X^{-1}$is defined[10] by $p \in [0,1] \rightarrow F_X^{-1}(p) = Inf\left\{x \in \overline{\mathbb{R}}, F_X(x) \geq p\right\}$, one gets $F_X^{-1}(0) = -\infty$, $F_X^{-1}(1) = \underset{s \in S}{Max} X(s)$, F_X^{-1} is non-decreasing and left-continuous. Accordingly, it is enough to know F_X^{-1} on $(0,1)$ to be completely informed on F_X^{-1}.

Let us also recall that $\forall X \in \mathbb{R}^S$, the mathematical expectation of X w.r.t. P, $E_P(X)$ is equal to $\int_0^1 F_X^{-1}(p)dp$.

Let us now come to the Choquet integral. First we recollect some definitions.

Definition 1 *υ is a capacity on $(S, \mathcal{A})$ if $\upsilon : A \in \mathcal{A} \rightarrow \upsilon(A) \in \mathbb{R}$, where $\upsilon(\varnothing) = 0$, $\upsilon(S) = 1$ and $(A, B) \in \mathcal{A}^2$ such that $A \subseteq B \Longrightarrow \upsilon(A) \leq \upsilon(B)$.*

Definition 2 *(Choquet integral) Let $X \in \mathbb{R}^S$ and υ a capacity on $\mathcal{A}$, the Choquet integral of X w.r.t. υ denoted $\int X d\upsilon$ is defined by $\int X d\upsilon = \int_{-\infty}^0 (\upsilon(X \geq t) - 1)dt + \int_0^{+\infty} \upsilon(X \geq t)dt$.*

So if we consider $X = x_1 A_1^* + ... + x_i A_i^* + ... + x_n A_n^*$, where the $A_i^{'}$s belong to $\mathcal{A}$ and form a partition of S, $x_1 < x_2 < ... < x_i < ... < x_n$ and A^* is the characteristic function of event A, i.e., $A^*(s) = 1$ if $s \in A$ and $A^*(s) = 0$ if $s \notin A$.

We obtain that $\int X d\upsilon = x_1(1 - \upsilon(X \geq x_2) + ... + x_i(\upsilon(X \geq x_i) - \upsilon(X \geq x_{i+1})) + ... + x_n \upsilon(X \geq x_n)$ or equally $\int X d\upsilon = x_1(1 - \upsilon(X > x_1) + ... + x_i(\upsilon(X > x_{i-1}) - \upsilon(X > x_i)) + ... + x_n \upsilon(X > x_{n-1})$.

We intend now to introduce suitable definitions of the cumulative distribution and quantile-function of an act X with respect to a capacity υ.

Let us recall:

Definition 3 *The dual capacity $\overline{\upsilon}$ of a capacity υ is defined by $\overline{\upsilon}(A) = 1 - \upsilon(A^C)$ $\forall A \in \mathcal{A}$.*

We now introduce the following new definitions.

Definition 4 *The cumulative distribution F_X^υ of X with respect to capacity υ is defined by $x \in \mathbb{R} \rightarrow F_X^\upsilon(x) = \overline{\upsilon}(X \leq x)$.*

The reader might be surprised that $F_X^\upsilon(x)$ would not be defined by $\upsilon(X \leq x)$; but this definition is in accordance with the probabilistic case since if υ equals a probability P then $F_X^\upsilon(x) = 1 - P(X > x) = P(X \leq x) = F_X^P(x)$. The fact that we do not choose $F_X^\upsilon(x)$ defined by $\upsilon(X \leq x)$ but by $\overline{\upsilon}(X \leq x)$ will be clear in the sequel, when defining the pseudo-inverse of F_X^υ and using it as in the probabilistic case for retrieving the Choquet integral.

Definition 5 *As in the probabilistic case, we define the quantile function $F_X^{\upsilon^{-1}}$ by $p \in [0,1] \rightarrow F_X^{\upsilon^{-1}}(p) = Inf\left\{x \in \overline{\mathbb{R}}, F_X^\upsilon(x) \geq p\right\}$.*

[10] To avoid confusion, it will be often useful here to write $F_X^{P^{-1}}$ instead of F_X^{-1}

Again, one gets $F_X^{\upsilon^{-1}}(0) = -\infty$, $F_X^{\upsilon^{-1}}(1) = \underset{s\in S}{Max} X(s)$, $F_X^{\upsilon^{-1}}$ is non-decreasing and left-continuous. Again $F_X^{\upsilon^{-1}}$ is completely defined by its values on $(0,1)$.

From this, we deduce:

Theorem 1 *$\forall X \in \mathbb{R}^S$: $\int X d\upsilon = \int_0^1 F_X^{\upsilon^{-1}}(p)dp$.*

Proof. *Let $X = x_1 A_1^* + ... + x_i A_i^* + ... + x_n A_n^*$, $A_i \in \mathcal{A}$, partition of S, $x_1 < ... < x_i < ... < x_n$. Let us denote $\alpha_i := \overline{\upsilon}(X \leq x_i)$.[11] It is straightforward to check that denoting $\alpha_0 = 0$, one gets $F_X^{\upsilon^{-1}}(p) = x_i$ for p belonging to $(\alpha_{i-1}, \alpha_i]$ for $i = 1, ..., n$. Therefore, writing F^{-1} instead of $F_X^{\upsilon^{-1}}$, for the sake of simplicity, one obtains:* $\int_0^1 F^{-1}(p)dp = \int_0^{\alpha_1} F^{-1}(p)dp + \int_{\alpha_1}^{\alpha_2} F^{-1}(p)dp + ... + \int_{\alpha_{i-1}}^{\alpha_i} F^{-1}(p)dp + ... + \int_{\alpha_{n-1}}^{\alpha n} F^{-1}(p)dp$

hence $\int_0^1 F^{-1}(p)dp = x_1(1 - \upsilon(X > x_1) + x_2(\upsilon(X > x_1) - \upsilon(X > x_2)) + ... + x_i(\upsilon(X > x_{i-1}) - \upsilon(X > x_i)) + ... + x_n \upsilon(X > x_{n-1})$

so $\int_0^1 F^{-1}(p)dp = \int X d\upsilon$ *QED* ■

3 Precautionary principle as a rule of choice with optimism on the upper tail and pessimism on the lower tail

Thus, according to our motivation, the DM faces uncertainty modeled by a convex capacity υ, i.e., a capacity satisfying the further requirement $\upsilon(A \cup B) + \upsilon(A \cap B) \geq \upsilon(A) + \upsilon(B)$, $\forall A, B \in \mathcal{A}$.

From the previous developments, it turns out that the DM will fix what she conceives as the lower tail and the upper tail, through her personal choice of $\alpha, \beta \in [0,1]$, where $\alpha \leq \beta$, and therefore will compute the value of $X \in \mathbb{R}^S$ through $I(X) = I_1(X) + I_2(X) + I_3(X)$, where: $I_1(X) = \int_0^\alpha F_X^{\upsilon^{-1}}(p)dp$; $I_2(X) = \int_\alpha^\beta F_X^{\pi^{-1}}(p)dp$ and $I_3(X) = \int_\beta^1 F_X^{\overline{\upsilon}-1}(p)dp$.[12]

First, we need to check that I satisfies the minimal requirements of consistency, which are monotonicity, i.e., $X \geq Y \implies I(X) \geq I(Y)$, and constant homogeneity, i.e., $I(a.S^*) = a$, $\forall a \in \mathbb{R}$.

Proposition 2 *I is monotone and constant homogeneous.*

Proof. *Let us confine ourselves to checking monotonicity, constant homogeneity being straightforward. Let $X \geq Y$, it is immediate that for any capacity υ, hence for any probability π, $\overline{\upsilon}(Y \leq t) \geq \overline{\upsilon}(X \leq t)$ $\forall t$ since $X \leq t \implies Y \leq t$ and $\overline{\upsilon}$ is monotone, so $F_X^\upsilon(t) \leq F_Y^\upsilon(t)$ $\forall t$, and consequently $F_X^{\upsilon^{-1}}(p) \geq F_Y^{\upsilon^{-1}}(p)$ $\forall p \in [0,1]$. Therefore by integrating w.r.t. p, one obtains $I(X) \geq I(Y)$* ■

[11] Note that $\alpha_n = 1$

[12] It is worth noting that $I(X) = E_\pi(X)$ if $\upsilon = \pi$ a probability; $I(X) = \int X d\upsilon$ if $\alpha = 1$ and $I(X) = \int X d\overline{\upsilon}$ if $\beta = 0$.

We need to check as desired that the DM is pessimistic with respect to outcomes in the lower tail and optimistic with respect to outcomes in the upper tail. Let us first make these statements precise.

Definition 6 *The DM is pessimistic with respect to the lower tail if she overestimates losses and underestimates gains in this tail with respect to her reference probability* π*, i.e., if* $I_1(X) \leq \int_0^{\alpha} F_X^{\pi^{-1}}(p)dp$.

Definition 7 *The DM is optimistic with respect to the upper tail if she underestimates losses and overestimates gains in this tail with respect to her reference probability* π*, i.e., if* $I_3(X) \geq \int_{\beta}^{1} F_X^{\pi^{-1}}(p)dp$.

Proposition 3 *The DM is pessimistic with respect to the lower tail and optimistic with respect to the upper tail.*

Proof. *We just confine to prove the pessimistic side; the proof is similar for the optimistic one.*

Let $p \in [0,1]$ *and recall that* $F_X^{\upsilon^{-1}}(p) = Inf\left\{x \in \overline{\mathbb{R}}, 1-\upsilon(X>x) \geq p\right\}$ *while* $F_X^{\pi^{-1}}(p) = Inf\left\{x \in \overline{\mathbb{R}}, 1-\pi(X>x) \geq p\right\}$*. Since* π *belongs to the core of* υ*, then* $\pi(X > x) \geq \upsilon(X > x)$*,* $\forall x \in \overline{\mathbb{R}}$*; therefore,* $1-\upsilon(X>x) \geq 1-\pi(X>x)$*,* $\forall x \in \overline{\mathbb{R}}$*, and consequently* $F_X^{\pi^{-1}}(p) \geq F_X^{\upsilon^{-1}}(p)$*,* $\forall p \in [0,1]$*. Hence* $\int_0^{\alpha} F_X^{\pi^{-1}}(p)dp \geq \int_0^{\alpha} F_X^{\upsilon^{-1}}(p)dp = I_1(X)$ *which completes the proof* ■

4 An application

Let $S = \{s_1, ..., s_i, ..., s_n\}$ and assume that the DM faces a situation of total uncertainty. In such a case, the DM might apply the Laplace's principle of Insufficient Reason and model a priori uncertainty through the uniform distribution $P_0 : P_0(\{s_i\}) = \frac{1}{n}, \forall i$. But ambiguity aversion might consistently lead her to envision the set of probability distributions in the core of υ, where υ is the $\mathcal{E}$-*contamination* of P_0, and $\mathcal{E}$ makes precise her ambiguity aversion: $\upsilon(A) = (1-\mathcal{E})P_0(A), \forall A \neq S$ and $\mathcal{E} \in [0,1]$.[13]

So in applying our model, the first step is to derive π. Clearly, $P_0 \in C(\upsilon)$, and since P_0 is the uniform distribution, it turns out that indeed $\pi = P_0$.

The purpose of this example being only to illustrate our model, we consider the simple case where $n = 100$, $\mathcal{E} = \frac{1}{2}$ (so we consider a DM with a high level of ambiguity aversion, which will mean a high level of pessimism with respect to the lower tail and a high level of optimism with respect to the upper tail), and where the lower and upper tails are consistently defined, respectively, through $\alpha = \frac{5}{100}$ and $\beta = \frac{95}{100}$.

We furthermore consider $X \in \mathbb{R}^S$ such that $X(I_i) = x_i$ with $x_1 < .... < x_i < ... < x_n$.

Let us first compute $I_2(X)$.

[13] Kopylov (2009) and Chateauneuf *et al.* (2010) axiomatize preferences represented by the $\mathcal{E}$-*contamination* models.

Note that $F_X^{\pi^{-1}}(\frac{i}{100}) = x_i$ for $i \in [1, 100]$ and let us recall that $F_X^{\pi^{-1}}(.)$ is left-continuous. It comes that $I_2(X) = \int_{5\%}^{6\%} x_6 dp + ... + \int_{94\%}^{95\%} x_{95} dp = \frac{x_6 + ... + x_{95}}{100}$.

Let us now turn to the computation of $I_1(X)$.

Recall that $F_X^{\upsilon}(x) = 1 - \upsilon(X > x)$, $\forall x \in \overline{\mathbb{R}}$, so $F_X^{\upsilon}(x_i) = 1 - (1 - \mathcal{E})\frac{n-i}{100}$ for $i \in [1, n]$.

Let us finally compute $I_3(X)$.

We have $F_X^{\overline{\upsilon}}(x) = \upsilon(X \leq x) = (1 - \mathcal{E})P(X \leq x)$, if $x < x_n$; hence $F_X^{\overline{\upsilon}}(x_i) = \frac{(1-\mathcal{E})i}{100}$, if $i < n$ so $F_X^{\overline{\upsilon}}(x_i) \geq \frac{95}{100} \iff i \geq 190$, if $i < 100$, which is impossible, henceforth $F_X^{\overline{\upsilon}^{-1}}(\frac{95}{100}) = x_n$ and $I_3(X) = \int_{95\%}^{100\%} x_n dp = \frac{5x_n}{100}$.

One finally obtains:

$I(X) = I_1(X) + I_2(X) + I_3(X) = \frac{5x_1}{100} + \frac{x_6 + ... + x_{95}}{100} + \frac{5x_n}{100}$.[14]

Note the application was dealing with the particular case of initial ambiguity consisting of an $\mathcal{E}$-*contamination* of the uniform distribution P_0. In such a case, the obtainment of the less diffuse π is immediate, since P_0 makes the prior. Indeed, in the more general case of uncertainty consisting of the core of a convex capacity υ, the obtainment of the less diffuse π in $C(\upsilon)$ is not totally immediate even if an efficient algorithm was proposed by Jaffray (1995).

5 A simpler model

If one wishes to overcome the use of Jaffray's algorithm in order to obtain a simpler model, one might consider first a credible probability measure P_0, and assume that the degree of pessimism with the lower tail is $\mathcal{E}_1$, while the degree of optimism with respect to the upper tail is $\mathcal{E}_2$.

Therefore, denoting υ_1 the $\mathcal{E}_1$-*contamination* of P_0 and υ_2 the $\mathcal{E}_2$-*contamination* of P_0, one might value act X through:

$I(X) = I_1(X) + I_2(X) + I_3(X)$, where

$I_1(X) = \int_0^{\alpha} F_X^{\upsilon_1^{-1}}(p)dp$; $I_2(X) = \int_{\alpha}^{\beta} F_X^{P_0^{-1}}(p)dp$ and $I_3(X) = \int_{\beta}^{1} F_X^{\overline{\upsilon}_2^{-1}}(p)dp$.

Such a model would also allow the expression of asymmetric pessimism and optimism through, for instance, $\mathcal{E}_1 \geq \mathcal{E}_2$.

It is straightforward to check that this new functional again satisfies the minimal requirement of monotonicity and constant homogeneity.

6 Concluding remarks

In this paper, we show a representation theorem in which an ambiguous act is evaluated by a functional defined through quantiles. We use a couple of quantiles that define an interval of events that the DM considers familiar, in some sense ordinary with respect to her experimented life, and two tails included extreme

[14] Note that this formula might appear meaningful, since in this simple case, due to strong pessimism for the lower tail and strong optimism for upper tail, the DM computes the value of X through the mathematical expectation of X with respect to the uniform distribution, by merely replacing outcomes in the lower tail with the worst outcome and outcomes in the upper tail with the best outcome.

events, that is, events with very small probabilities of occurring but very large consequences both positive (windfall gains) and negative (catastrophic losses). In this way, we are able to take into account not only asymmetric attitude with respect to ambiguity on extreme events (optimism with respect to windfall gains and pessimism with respect to catastrophic events) but also the DM's attitude to consider entropy as a rule of inference when information is ambiguous and scanty.

Our approach is consistent and intuitive. It originates a probability distribution on the possible consequences of a given act that has an interesting shape: it is a platykurtic (fat-tailed) function. It is straightforward to extend our representation to financial markets where phenomena such as implicit volatility smiles on stock options (volatility varies across state and time to maturity) are extensively documented. Finally, our representation theorem allows one to give operational content to the precautionary principle by defining a rank among alternative acts that combines conservative and dissipative behavior with the application of the principle of insufficient reason (one egg in each box) given the reliability of the probability distributions and overcomes the failure of the full conservative measure, e.g., maxmin decisional rule.

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